ALGEBRAIC CURVES
EXERCISE SHEET 7

Unless otherwise specified, k is an algebraically closed field.

Exercise 1.

Let X be a topological space and $Y \subseteq X$ a topological subspace. Show the following assertions:

(1) $dim(Y) \leq dim(X)$.
(2) If $(U_i)_{1\leq i\leq n}$ is an open cover of X, then $dim(X) = sup_{1\leq i\leq n}(dim(U_i))$.
(3) Find an example of $Y \subseteq X$ such that Y is open and dense in X and $dim(Y) < dim(X)$. (Hint: Consider a topological space consisting of two points, with only one of them being closed).
(4) If X is irreducible, has finite dimension and Y is closed, then $dim(Y) = dim(X) \Leftrightarrow Y = X$.

Exercise 2.

Let V, W be algebraic varieties and $P \in V$. Show the following assertions:

(1) $dim(V) = 0 \Leftrightarrow$ V is finite. (Hint: reduce to the affine case via an open affine cover of V).
(2) If V and W are affine, then $dim(V \times W) = dim(V) + dim(W)$. (Hint: by Noether normalization, the transcendence degree of $k(V)$ is also the cardinality of a maximal subset of algebraically independent elements of $\Gamma(V)$).

Exercise 3. *

Let $Y = \{(t^3, t^4, t^5),\ t \in k\} \subseteq \mathbb{A}^3_k$. We denote by $ht(I)$ the height of an ideal I.

(1) Show that Y is a subvariety of $\mathbb{A}^3_k$ and compute $I(Y)$.
(2) Compute $r = ht(I(Y))$. Show that $I(Y)$ cannot be generated by r elements. (Hint: Show that $dim_k(I(Y)/(x,y,z)I(Y)) \geq 3$).

Exercise 4.

We denote by $k(V)$ the field of fractions of an algebraic variety V. Let $n \geq 1$.

(1) Compute $k(\mathbb{P}_k^n)$.
(2) Let $V_1 = V(y^2 - x^3) \subseteq \mathbb{A}_k^2$. Show that $k(V_1) \simeq k(\mathbb{P}_k^1)$. Is V_1 isomorphic to $\mathbb{P}_k^1$?
(3) Let $V_2 = V_P(x_1x_2 - x_3x_4) \subseteq (\mathbb{P}_k^3$. Show that $k(V_2) \simeq k(\mathbb{P}_k^2)$. Is V_2 isomorphic to $\mathbb{P}_k^2$? (Hint: You may assume that any two curves in $\mathbb{P}_k^2$ intersect).

Exercise 5.

Let $X = \mathbb{P}_k^2 \setminus \{x\}$ the complement of a point $x \in \mathbb{P}_k^2$.

(1) Compute $O(X)$ and $k(X)$.
(2) Show that X is not quasi-affine nor projective.